Worcester County Mathematics League

Freshman/JV Meet 2
December 14, 2016

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 14, 2016

Round 1: Algebraic Word Problems

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Ann runs twice as fast Ed walks. In 30 minutes, Ann can run 6 kilometers farther than Ed can walk. How fast does Ed walk?

2. A train one mile long enters a tunnel which is one mile long at 1 pm. If the train is moving at a constant rate of 20 mph, at what time will the train clear the tunnel completely?

3. Oliver invested part of $15,000 at 6% and the remainder at 4%. If he had instead doubled his investment at 6% he would have increased his year end return by $120. How much did Oliver originally invest at 6%?

ANSWERS

(1 pt.) 1. ______________________ km/hr

(2 pts.) 2. ________ : ________________ pm

(3 pts.) 3. $ ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 14, 2016

Round 2: Number Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Every positive integer has a unique prime factorization. In general, this factorization is presented as $p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_k^{a_k}$, where each p_i is a positive prime number and each a_i is a positive integer.

Write the prime factorization of 620.

2. If x is a base 10 number, then x_2 represents the same number in base 2. Compute the least common multiple of 101101_2 and 202200_3 in base 10.

3. Find the sum of all natural numbers that divide 3087 without remainder.

ANSWERS

(1 pt.) 1. ______________________________

(2 pts.) 2. ______________________________

(3 pts.) 3. ______________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 14, 2016

Round 3: Operations on Numerical Fractions, Decimals, Percents, and Percentage Word Problems

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Simplify: $\dfrac{\frac{2}{3}+\frac{4}{6}+\frac{6}{9}}{\frac{3}{2}+\frac{6}{4}+\frac{9}{6}+\frac{12}{8}}$

2. Julio has a collection of 20 miniature cars and trucks. Cars make up 40% of the collection. If Julio adds only cars to his collection, how many cars must he add to make the collection 75% cars?

3. Let $x = 0.3\overline{8}$ and $y = 0.\overline{38}$. Express $\frac{x}{y}$ as a simplified fraction (or mixed number) using only integers for the numerator and denominator.

ANSWERS

(1 pt.) 1. ________________

(2 pts.) 2. ________________ cars

(3 pts.) 3. ________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 14, 2016

Round 4: Set Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

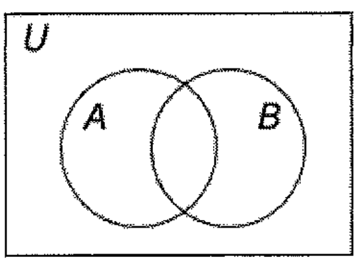
1. If A^C denotes the complement of set A and U is the universal set, shade $(A \cap B)^C \cap A$ below.

2. A set containing $k+1$ elements has 8 more subsets than a set containing k elements. What is k?

3. For three sets A, B and C, define $f(A,B,C) = (A \cap B) \cup (B^C \cap C^C)$. In the region below, if U is the universal set, shade in $f(X, Y, Z) \cap f(Y, Z, X) \cap f(Z, X, Y)$. (Note: Scrap paper for this problem will be provided)

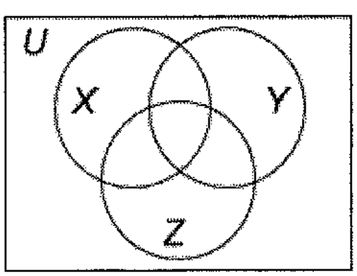
ANSWERS

(1 pt.) 1.



(2 pts.) 2. ____________________

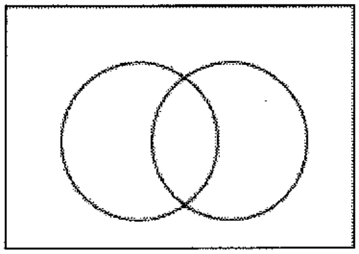
(3 pts.) 3.

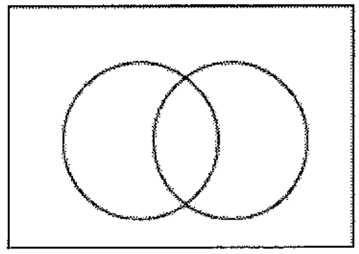


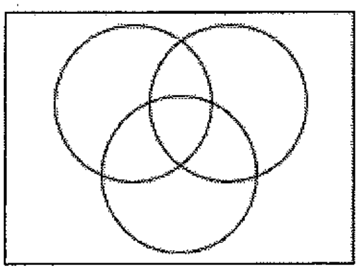
WORCESTER COUNTY MATHEMATICS LEAGUE

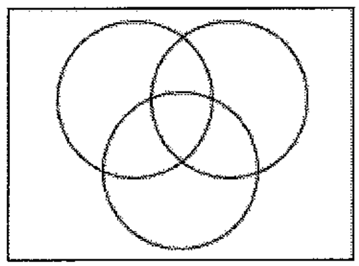
Freshman Meet 2 - December 14, 2016

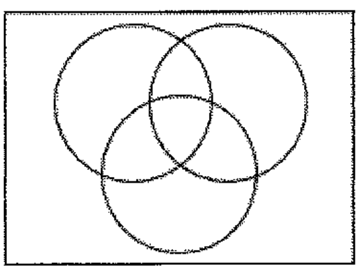
Round 4: SCRAP PAPER

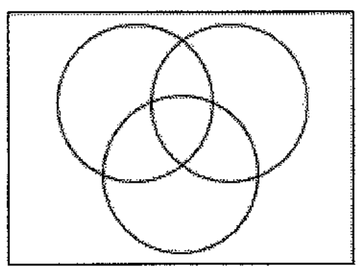


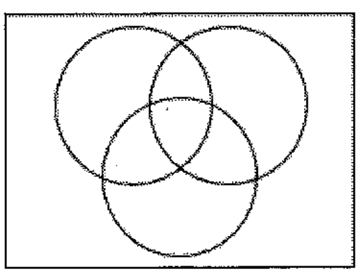


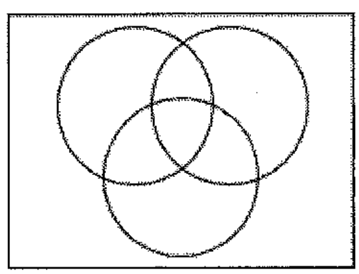












WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 14, 2016

Team Round

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 points each)

APPROVED CALCULATORS ALLOWED

1. There are three consecutive even integers such that $\frac{1}{2}$ of the first added to $\frac{3}{8}$ of the second is equal to $\frac{5}{6}$ of the third. What is the largest of these integers?

2. Find the smallest positive integer k such that $90k$ is a perfect cube.

3. The original price of an item is reduced by 10%. By what percentage must this new price be increased so that the resulting price is 8% more than the original price?

4. There are 42 students at a school and everyone plays at least one sport. 19 play basketball. 17 play baseball. 8 play basketball and football. 7 play basketball and baseball. 3 play baseball and football. One student plays all three sports. How many students play only football?

5. Solve and write the solution on the number line: $|x-2| < 3$.

6. Compute n if $\left(10^{12}+25\right)^2-\left(10^{12}-25\right)^2=10^n$.

7. Compute:

	3 Weeks	3 Days	3 Hours	3 Minutes	35 Seconds
-	1 Week	5 Days	5 Hours	5 Minutes	5 Seconds

8. The second side of a triangle is $\frac{1}{3}$ longer than the first side. If the third side is 50% longer than the second side and the perimeter is 39 inches, what is the length of the triangle's shortest side?

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 14, 2016 ANSWER KEY

Round 1:

1. 12 (Shepherd Hill)
2. 1:06 (Bromfield)
3. 6000 (Tahanto)

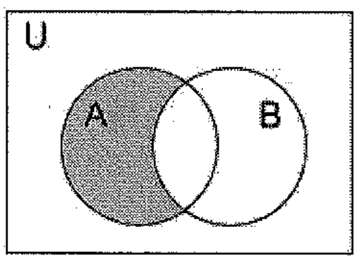
Round 2:

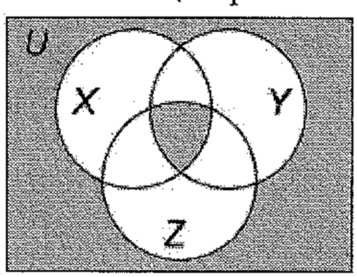
1. $2^2 \times 5 \times 31$ or $2 \times 2 \times 5 \times 31$ (Southbridge)
2. 2790 (Quaboag)
3. 5200 (Westborough)

Round 3:

1. $\frac{1}{3}$ or $0.\overline{3}$ or $0.\overline{333}$ (West Boylston)
2. 28 (Bromfield)
3. $\frac{77}{76}$ or $1\frac{1}{76}$ (Worcester Academy)

Round 4:

1. 

(Doherty)
2. 3 (Shepherd Hill)
3. 

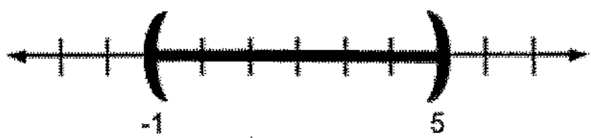
(Assabet Valley)

TEAM Round

1. 66 (Hudson)
2. 300 (Worcester Academy)
3. 20 (Worcester Academy)
4. 13 (Millbury)
5. 

(Shepherd Hill)

--OR--



6. 14 (Burncoat)
7. 1 4 21 58 30 (Assabet Valley)
8. 9 (Burncoat)

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 14, 2016
Team Round Answer Sheet

1. ______________________

2. ______________________

3. ______________________ %

4. ______________________ students

5. 

6. ______________________

7. ____ Weeks ____ Days ____ Hours ____ Minutes ____ Seconds

8. ______________________ units

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 14, 2016 - SOLUTIONS

Round 1: Algebraic Word Problems

1. Ann runs twice as fast Ed walks. In 30 minutes, Ann can run 6 kilometers farther than Ed can walk. How fast does Ed walk?

Solution: Let x be Ed's walking speed which means Ann's running speed is $2x$. We have

$$\frac{1}{2}(2x-x)=6$$
$$(2x-x)=12$$
$$x=12.$$

2. A train one mile long enters a tunnel which is one mile long at 1 pm. If the train is moving at a constant rate of 20 mph, at what time will the train clear the tunnel completely?

Solution: Since we are interested in the time it takes for the train to completely clear the tunnel, we equivalently want to know how long it takes the front of the train to travel 2 miles. We know that Time = Distance ÷ Speed. Therefore we have

$$\text{Time} = 2 \div 20 = \tfrac{1}{10} \text{ hours.}$$

One tenth of an hour is six minutes, so the train will clear the tunnel completely at 1:06 pm.

3. Oliver invested part of $15,000 at 6% and the remainder at 4%. If he had instead doubled his investment at 6% he would have increased his year end return by $120. How much did Oliver originally invest at 6%?

Solution: Let x be the amount that Oliver invested a 6%. Therefore, we know that he invested $15000-x$ at 4%. From the given information we have that

$$0.06\times 2x+0.04\times(15000-2x)=0.06\times x+0.04\times(15000-x)+120$$
$$0.06\times(2x-x)+0.04\times[(15000-2x)-(15000-x)]=120$$
$$0.06x+0.04\times[-x]=120$$
$$0.02x=120$$
$$x=6000.$$

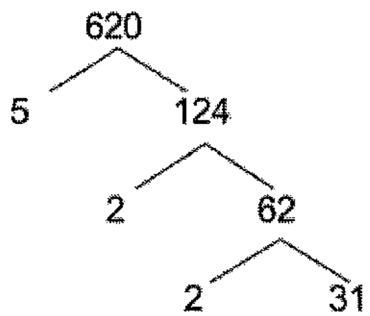
Round 2: Number Theory

1. Every positive integer has a unique prime factorization. In general, this factorization is presented as $p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_k^{a_k}$, where each p_i is a positive prime number and each a_i is a positive integer.

Write the prime factorization of 620.

Solution 1: We have that $620 = 2 \times 310 = 2^2 \times 155 = 2^2 \times 5 \times 31$.

Solution 2: We have that



Therefore, we have that $620 = 2^2 \times 5 \times 31$.

2. If x is a base 10 number, then x_2 represents the same number in base 2. Compute the least common multiple of 101101_2 and 202200_3 in base 10.

Solution 1: We begin by converting both numbers into base 10. We have that

$$101101_2 = 1 \cdot 2^0 + 1 \cdot 2^2 + 1 \cdot 2^3 + 1 \cdot 2^5$$
$$101101_2 = 1 + 4 + 8 + 32 = 45$$

$$202200_3 = 2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^5$$
$$202200_3 = 18 + 54 + 486 = 558$$

To compute the LCM, we need to break both numbers into their prime factorizations

$$45 = 3^2 \times 5$$
$$558 = 2 \times 279 = 2 \times 3 \times 93 = 2 \times 3^2 \times 31$$

Therefore, the LCM = $2 \times 3^2 \times 5 \times 31 = 558 \times 5 = 2790$.

Solution 2: Begin by converting both numbers into base 10. We have that

$$101101_2 = 1\cdot 2^5 + 1\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2^0$$
$$101101_2 = 32 + 8 + 4 + 1 = 45$$

$$202200_3 = 2\cdot 3^5 + 2\cdot 3^3 + 2\cdot 3^2$$
$$202200_3 = 2\cdot 243 + 2\cdot 27 + 2\cdot 9$$
$$202200_3 = 486 + 54 + 18 = 558$$

We can now compute the desired LCM by writing

$$\begin{array}{r|cc} 3 & 45 & 558 \\ 3 & 15 & 186 \\ \hline & 5 & 62 \end{array}$$

We now have that the LCM is simply $3\times 3\times 5\times 62 = 9\times 310 = 2790$.

3. Find the sum of all natural numbers that divide 3087 without remainder.

Solution 1 (Prime Factorization): To begin, break 3087 into its prime factorization:

$$3087 = 3\times 1029 = 3^2\times 343 = 3^2\times 7^3.$$

The natural numbers which divide 3087 without remainder all must take the form $3^a\times 7^b$, where $0\le a\le 2$ and $0\le b\le 3$. For example, one such natural number would be $3^2\times 7^0 = 9$. The sum of all such natural numbers is precisely equal to

$$(3^0 + 3^1 + 3^2)(7^0 + 7^1 + 7^2 + 7^3)$$
$$(1+3+9)(1+7+49+343)$$
$$(13)(400) = 5200.$$

Solution 2 (Brute Force): We can solve the problem by enumerating all the positive factors of 3087. We have that 3087 can be factored in the following ways

1	×	3087
3	×	1029
7	×	441
9	×	343
21	×	147
49	×	63

The sum of the factors in the left hand column is 90 and the sum of the factors in the right hand column is 5110. Together, we have that the sum of all the positive factors of 3087 is 5200.

Round 3: Operations on Numerical Fractions, Decimals, Percents, and Percentage Word Problems

1.Simplify: $\dfrac{\frac{2}{3}+\frac{4}{6}+\frac{6}{9}}{\frac{3}{2}+\frac{6}{4}+\frac{9}{6}+\frac{12}{8}}$

Solution: Notice first that $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}$ and that $\frac{3}{2}+\frac{6}{4}+\frac{9}{6}+\frac{12}{8}$. This lets us write that

$$\frac{\frac{2}{3}+\frac{4}{6}+\frac{6}{9}}{\frac{3}{2}+\frac{6}{4}+\frac{9}{6}+\frac{12}{8}} = \frac{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}}{\frac{3}{2}+\frac{3}{2}+\frac{3}{2}+\frac{3}{2}} = \frac{3\times\frac{2}{3}}{4\times\frac{3}{2}} = \frac{2}{2\times 3} = \frac{1}{3}.$$

2. Julio has a collection of 20 miniature cars and trucks. Cars make up 40% of the collection. If Julio adds only cars to his collection, how many cars must he add to make the collection 75% cars?

Solution: First we note that Julio has $\frac{4}{10}\times 20 = 8$ cars in his original collection. Now let x be the number of cars Julio needs to add to his collection so that it becomes 75% cars. We have that

$$\begin{aligned}
&\frac{75}{100} = \frac{8+x}{20+x} \\
&75\times(20+x) = 100\times(8+x) \\
&1500+75x = 800+100x \\
&700 = 25x \\
&x = 28.
\end{aligned}$$

3. Let $x = 0.3\overline{8}$ and $y = 0.\overline{38}$. Express $\frac{x}{y}$ as a simplified fraction (or mixed number) using only integers for the numerator and denominator.

Solution: Begin by converting the numerator into an integer fraction. Let $x = 0.3\overline{8}$. We have that

$$\begin{aligned}
&100x = 38.\overline{8} \\
&100x - 10x = 38.\overline{8} - 3.\overline{8} \\
&90x = 35 \\
&x = \tfrac{35}{90} = \tfrac{7}{18}.
\end{aligned}$$

Now let $y = 0.\overline{38}$. We have that

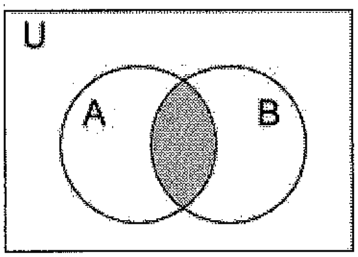
$$\begin{aligned}
100y &= 38.\overline{38} \\
100y - y &= 38.\overline{38} - 0.\overline{38} \\
99y &= 38 \\
y &= \tfrac{38}{99}
\end{aligned}$$

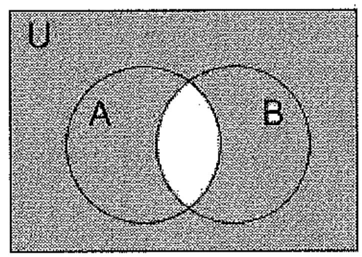
Therefore, we have that $\frac{0.3\overline{8}}{0.\overline{38}} = \frac{x}{y} = \frac{7}{18} \div \frac{38}{99} = \frac{7}{18} \div \frac{99}{38} = \frac{7}{2} \div \frac{11}{38} = \frac{77}{76}$.

Round 4: Set Theory

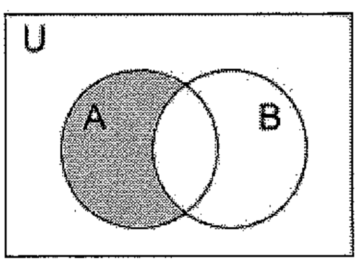
1. If A^C denotes the complement of set A and U is the universal set, shade $(A \cap B)^C \cap A$ below.

Solution: We have that

$A \cap B =$ 

$(A \cap B)^C =$ 

And finally,

$(A \cap B)^C \cap A =$ 

2. A set containing $k+1$ elements has 8 more subsets than a set containing k elements. What is k?

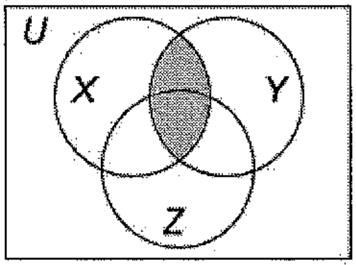
Solution: We know that a set containing k elements has precisely 2^k subsets. To solve this problem, we will guess different values of k and see if we can find a k such that $2^{k+1}-2^k=8$.

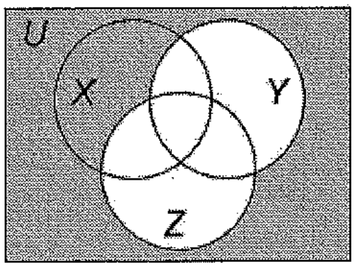
k	2^{k+1}	2^k	$2^{k+1}-2^k$
1	4	2	2
2	8	4	4
3	16	8	8
4	32	16	16

We can see in the table that the guess of $k=3$ delivers the desired outcome. Therefore, we conclude that $k=3$.

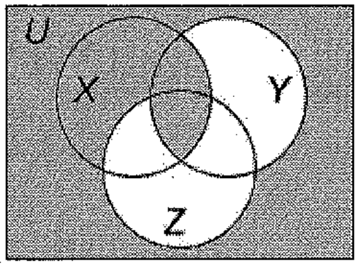
3. For three sets A, B and C, define $f(A,B,C)=(A\cap B)\cup(B^C\cap C^C)$. In the region below, if U is the universal set, shade in $f(X, Y, Z)\cap f(Y, Z, X)\cap f(Z, X, Y)$. (Note: Scrap paper for this problem will be provided)

Solution: Start with the first term of the desired expression, $f(X, Y, Z)$. We have that

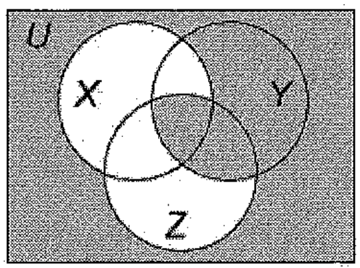
$(X\cap Y)=$ 

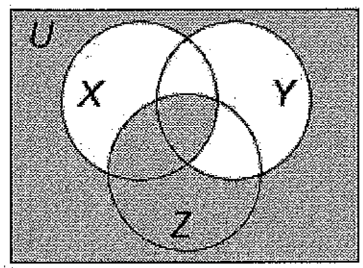
$(Y^C\cap Z^C)=$ 

And therefore we have that,

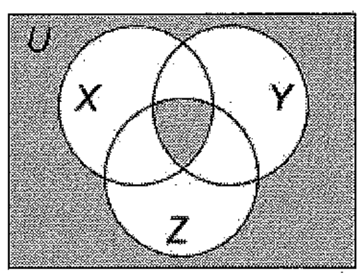
$f(X, Y, Z) =$ 

Notice now that the other terms in the desired expression almost identical to the first term but with the arguments slightly reordered. In the second term, $f(Y, Z, X)$, we see that compared to the first term Y has now taken the place of X, Z has now taken the place of Y and X has now taken the place of Z. Using this same logic, we can quickly draw

$f(Y, Z, X) =$ 

and $f(Z, X, Y) =$ 

We are looking for the intersection of the three drawings above. The only areas which are shaded in all three drawings are the center and the exterior:

$f(X, Y, Z) \cap f(Y, Z, X) \cap f(Z, X, Y) =$ 

Team Round

1. There are three consecutive even integers such that $\frac{1}{2}$ of the first added to $\frac{3}{8}$ of the second is equal to $\frac{5}{6}$ of the third. What is the largest of these integers?

Solution: Let x be the largest of the three integers. We have that

$$\frac{1}{2}(x-4)+\frac{3}{8}(x-2)=\frac{5}{6}x$$
$$\frac{1}{2}x-2+\frac{3}{8}x-\frac{3}{4}=\frac{5}{6}x$$
$$\frac{12}{24}x-\frac{48}{24}+\frac{9}{24}x-\frac{18}{24}=\frac{20}{24}x$$
$$12x-48+9x-18=20x$$
$$x=48+18=66.$$

2. Find the smallest positive integer k such that $90k$ is a perfect cube.

Solution: Begin by breaking 90 into its prime factorization: $90=3^2\times 10=2\times 3^2\times 5$. In order for $90k$ to be a perfect cube, all of its prime factors must be at a power which is a multiple of 3. The smallest k that achieves this will raise the power of each factor of 90 exactly to 3.

For example, take the single factor of 2 in the prime factorization of 90. We need to multiply this by 2^2 in order to give the desired result of 2^3. Likewise, we need to multiply the factor of 3^2 in the factorization of 90 by 3 in order to give 3^3. And finally, we need to multiply the factor of 5 by 5^2 in order to give 5^3.

Therefore, we have that $k=2^2\times 3\times 5^2=4\times 3\times 25=3\times 100=300$.

3. The original price of an item is reduced by 10%. By what percentage must this new price be increased so that the resulting price is 8% more than the original price?

Solution 1: Let x be the original price of the item and let y be the necessary percentage increase. We have that

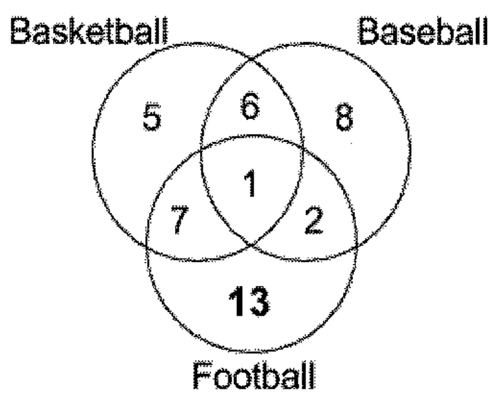
$$(0.90x)(1+y)=1.08x$$
$$1+y=\frac{1.08x}{0.9x}$$
$$1+y=\frac{1.08x}{0.9x}=\frac{1.08}{0.9}=1.2$$
$$y=0.2=20\%$$

Solution 2: Assume that the item costs \$100. A 10% price reduction would make the price of the item \$90. Since \$108 is 8% more than the original price of \$100, we need to increase the new reduced price of the good by \$18. \$18 is precisely 20% of \$90.

4. There are 42 students at a school and everyone plays at least one sport. 19 play basketball. 17 play baseball. 8 play basketball and football. 7 play basketball and baseball. 3 play baseball and football. One student plays all three sports. How many students play only football?

Solution: We can draw a Venn diagram of this situation as follows

We start by filling in that only one student plays all three sports.



Next, since 8 students play basketball and football, and one student plays all three sports, we can fill in the section for "Only plays football and basketball" with a 7.

Similarly, since 7 students play basketball and baseball, and one student plays all three sports, we can fill in the section for "Only plays baseball and basketball" with a 6.

Moreover, since 3 students play football and baseball, and one student plays all three sports, we can fill in the section for "Only plays baseball and football" with a 2.

Now since we are given that 19 students play basketball, we can fill in the section for "Only plays basketball" with $19-7-1-6=5$.

And since we are given that 17 students play baseball, we can fill in the section for "Only plays baseball" with $17-6-1-2=8$.

Finally to compute the number of students who only play football, we subtract all other numbers in the venn diagram from the total of 42 students:

$$42-5-6-8-7-1-2=13.$$

5. Solve and write the solution on the number line: $|x-2| < 3$.

Solution: This inequality translates into the two following inequalities:

$$x-2 > -3 \quad \text{and} \quad x-2 < 3$$
$$-3 < x-2 \quad \text{and} \quad x-2 < 3$$
$$-1 < x \quad \text{and} \quad x < 5$$

We graph this as

6. Compute n if $\left(10^{12}+25\right)^2-\left(10^{12}-25\right)^2=10^n$.

Solution 1 (Difference of Squares): Notice that we have a difference of squares in this equation. That means we can write

$$\left(10^{12}+25\right)^2-\left(10^{12}-25\right)^2=10^n$$
$$\left[\left(10^{12}+25\right)+\left(10^{12}-25\right)\right]\left[\left(10^{12}+25\right)-\left(10^{12}-25\right)\right]=10^n$$
$$\left[2\times 10^{12}\right][50]=10^n$$
$$\left[10^{12}\right][2\times 50]=10^n$$
$$\left[10^{12}\right][10^2]=10^n$$
$$10^{14}=10^n$$
$$n=14.$$

Solution 2 (Foiling): Begin by foiling the two terms on the left hand side. We get

$$\left(10^{12}+25\right)^2-\left(10^{12}-25\right)^2=10^n$$
$$\left(10^{12\times 2}+2\times 25\times 10^{12}+25^2\right)-\left(10^{12\times 2}-2\times 25\times 10^{12}+25^2\right)=10^n$$
$$10^{12\times 2}-10^{12\times 2}+25^2-25^2+2\times 25\times 10^{12}+2\times 25\times 10^{12}=10^n$$
$$4\times 25\times 10^{12}=10^n$$
$$100\times 10^{12}=10^n$$
$$10^2\times 10^{12}=10^n$$
$$10^{14}=10^n$$
$$n=14.$$

7. Compute:

	3 Weeks	3 Days	3 Hours	3 Minutes	35 Seconds
-	1 Week	5 Days	5 Hours	5 Minutes	5 Seconds

Solution 1: Work from right to left.

35 seconds - 5 seconds = 30 seconds.

We cannot directly compute 3 minutes - 5 minutes, so we need to borrow an hour's worth of minutes from the hours column. This gives 63 minutes - 5 minutes = 58 minutes.

Similarly, we cannot directly compute 2 hours - 5 hours, so we need to borrow one day's worth of hours from the days column. This gives 26 hours - 5 hours = 21 hours.

Moreover, we cannot directly compute 2 days - 5 days, so we need to borrow one week's worth of days from the weeks column. This gives 9 days - 5 days = 4 days.

Finally, we are left with 2 weeks - 1 week = 1 week.

Solution 2 (Visual):

		+7	+24		
	2	2	2	+60	
	~~3~~ Weeks	~~3~~ Days	~~3~~ Hours	3 Minutes	35 Seconds
—	1 Week	5 Days	5 Hours	5 Minutes	5 Seconds
	1 Week	4 Days	21 Hours	58 Minutes	30 Seconds

8. The second side of a triangle is $\frac{1}{3}$ longer than the first side. If the third side is 50% longer than the second side and the perimeter is 39 inches, what is the length of the triangle's shortest side?

Solution: Let x be the length of the first side of the triangle. We have that the second side has a length of $(1+\frac{1}{3})x = \frac{4}{3}x$. We have that the third side has a length $(1+0.5)(\frac{4}{3}x) = \frac{3}{2} \times \frac{4}{3}x = 2x$.

Since the triangle has a total perimeter of 39 inches, we can write

$$x + \frac{4}{3}x + 2x = 39$$
$$\frac{3}{3}x + \frac{4}{3}x + \frac{6}{3}x = 39$$
$$\frac{13}{3}x = 39$$
$$x = 9.$$